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INTRODUCTION

In the following discussion the unicursal quartic is regarded from two points of view. Chapter I treats of the curve in its correspondence to a conic section through a quadratic reciprocal transformation. This leads to an interesting classification of unicursal quartics and affords a convenient and ready method for determining the form of the curve. Incidentally, it brings to light a geometrical application of the well known "Group of Four." In Chapter II the curve is defined as the locus of intersection of corresponding rays of two projective pencils of the second order. This develops properties of the curve not readily obtained in the other treatment. The discussion shows that the two definitions are not independent, but that each is supplementary to the other.

CHAPTER I

THE UNICURSAL QUARTIC IN CORRESPONDENCE TO A CONIC SECTION

1. *A Quadratic Reciprocal Transformation.*—The following well known geometrical construction of the general quadratic reciprocal transformation is of fundamental importance for the study of the unicursal quartic. Consider in the plane any two conic sections α and α' and a point P. The polars p and p' of P with respect to α and α', respectively, will intersect in a point P'. The polars of P' must likewise intersect in P. To P corresponds P', and vice versa. We say that P and P' are conjugate points of the transformation. By correlating all such pairs of conjugate points an involutory transformation is established between the points of the plane.

2. *Theorem.—As a point P describes a straight line, its conjugate point P' generates a conic section as the locus of the intersection of corresponding rays of two projective pencils of the first order.*

As P describes a straight line l, the polars p and p' of P with respect to α and α' describe projective pencils of rays L and L' of the first order.

3. *Theorem.—As a point P describes a conic its conjugate point P' generates a unicursal quartic, as the locus of the intersection of corresponding rays of two projective pencils of the second order.*

As P describes a conic γ, its polars p and p' with respect to α and α', respectively, describe projective pencils of rays κ and κ' of the second order. To determine the degree of the locus of P', the conjugate point of P, that is, the locus of the point of intersection of corresponding rays of the two projective pencils κ and κ' of the second order, cut it with a straight line l. To l corresponds a conic λ. The points in which l intersects the locus of P' correspond to the points in which λ intersects γ, the locus of P. But λ and γ can intersect in at most four points. The locus of P' is, therefore, a quartic. In establishing a one-to-one correspondence between the points of the quartic and a conic section, we have shown that the quartic is unicursal.

4. *Theorem.—Any curve of degree n is transformed into a curve of degree $2n$ as the locus of the intersection of correspondings rays of two projective pencils each of order n.*

As a point P describes a curve of degree n, its polars p and p' with respect to α and α', respectively, generate projective pencils of rays each of order n. To determine the degree of the locus of the point of intersection of corresponding rays, that is, of the point P', conjugate to P, we proceed as before in § 3, cutting the locus of P' with a straight line l. To l corresponds a conic λ. The intersections of l and the locus of P' correspond to the intersections of λ and the locus of P. But of these there can be at most $2n$.

5. *Theorem.—Any point on a side of the self-polar triangle of α and α' corresponds to the opposite vertex and vice versa.*

For if P lies on a side s_1 of the triangle $S_1\ S_2\ S_3$, self-polar with respect to α and α', its polars p and p' must meet in S_1, the vertex opposite to s_1.

6. *Theorem.—A curve of degree n corresponds by the above transformation to a curve of degree $2n$ having the vertices of the self-polar triangle as n-fold points.*

For from § 5 it follows that to each intersection of the curve of degree n described by P with a side of the self-polar triangle corresponds the opposite vertex as a point of the curve described by P'.

In special cases the curve of P may pass through a vertex of the self-polar triangle. The curve of P' then degenerates, the opposite side of the triangle appearing as a part of the curve. In general, a curve of degree n which passes in all k times through the vertices of the self-polar triangle goes into a degenerate curve of degree $2n$, which contains the sides of the triangle counted k times and a curve of degree $2n - k$. In particular, a line l passing through a vertex of the triangle corresponds to a line l' passing through the same vertex.

7. *The Quartics $Q\alpha$ and $Q\alpha'$.*—To the conic α (or α') itself corresponds a quartic $Q\alpha$ (or $Q\alpha'$) which passes through the four intersections of α and α', and also through the four points of contact on α' (or α) of the tangents common to α and α'. These curves can be traced at once for any given conics α and α'. They are, therefore, of great assistance in the construction of any curve C', for to the intersections of C (the corresponding curve) and $Q\alpha$ (or $Q\alpha'$) correspond the intersections of C' and α (or α'). These points of C' on α (or α') are the points of contact on α (or α') of one of the tangents drawn to α (or α') from the intersections of C and $Q\alpha$ (or $Q\alpha'$).

8. *Theorem.—Conjugate points P and P' of the transformation project to any vertex of the self-polar triangle in an involution of rays.*

This follows from § 6, where it was shown that to a line l passing through a vertex of the triangle corresponds a line l' passing through the same vertex, and vice versa. From this we see that it is possible to regard P' as the point determined by the two rays which correspond to the rays in which a point P projects to two arbitrary involution centers, S_1 and S_2. The line $S_1\ S_2$ will correspond to some line s_2 at S_1 and to a line s_1 at S_2, $s_1\ s_2$ determining the point S_3. $S_1\ S_2\ S_3$ is then a singular triangle of which each vertex corresponds to the opposite side and vice versa. The quadratic involutory transformation can be completely worked out from this point of view.*

9. *Theorem.—The double or focal rays of the involutions at the vertices of the self-polar triangle are in each case the two common chords of α and α' which intersect at that vertex.*

If this is true as P moves along c, a chord common to α and α', P' must also describe c. The four intersections of α and α' are the self-corresponding, or invariant, points of the transformation. c is then a line passing through a vertex of the self-polar triangle and through two invariant points, I_1 and I_2. It must then correspond to a line passing through the same vertex and through I_1 and I_2, that is, to itself.

If α and α' intersect in four real points, the double or focal rays are all real, that is, the involutions at the three vertices of the self-polar triangle are all hyperbolic; if they intersect in four imaginary points, at two of the three real vertices the involution is elliptic, at the third it is hyperbolic; if they intersect in two real and two imaginary points, the involution at the one real vertex is hyperbolic.

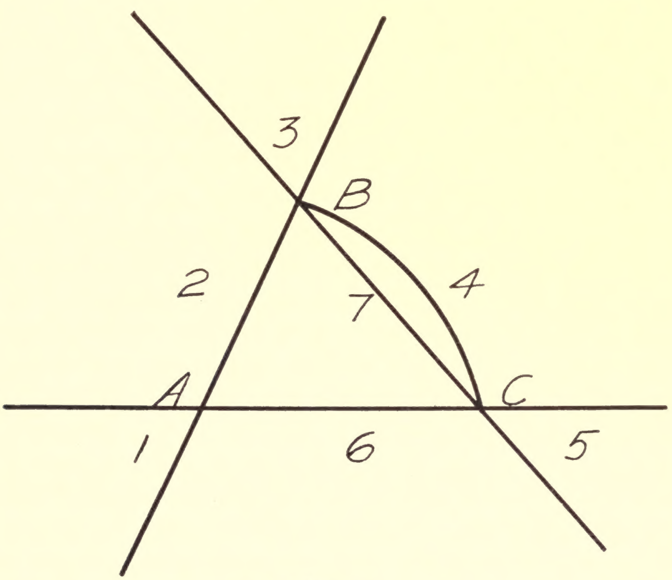

Fig. 1

* See D. N. Lehmer, "On the Combination of Involutions," *Am. Math. Mo.*, vol. 18, no. 3 (March, 1911).

10. *The Form of the Curve.*—If we regard the plane as divided into seven regions by the self-polar or singular triangle, we can, knowing the character of the involutions at the vertices of the triangle, determine in what region a point P' must lie that is conjugate to any given point P. Number the regions 1, 2, 3, 4, 5, 6, 7, and denote the vertices A, B, C, as in figure 1. If the involutions are all hyperbolic, a ray projected from a point P in 1 corresponds at A to a ray passing through the regions 1, 7, and 4; at B to a ray passing through the regions 2, 4, and 1 or 5; at C to a ray passing through the regions 6, 4, and 1 or 3. Any two of these three rays are sufficient to determine P' as a point in 4 or 1. In the same manner we can locate P' for any other point P in the plane, whether the involutions are all hyperbolic, or two are elliptic. If the involution is hyperbolic at A, B, and C, we find that:

P in 1 or 4 projects to a ray

$$\left.\begin{array}{l}\text{at A passing through 4, 7 and 1}\\ \text{at B passing through 4, 2 and 1 or 5}\\ \text{at C passing through 4, 6 and 1 or 3}\end{array}\right\} P' \text{ is, therefore, in 4 or 1}$$

P in 2 or 5

$$\left.\begin{array}{l}\text{A — 2, 6, 3 or 5}\\ \text{B — 2, 4, 1 or 5}\\ \text{C — 2, 7, and 5}\end{array}\right\} P' \text{ in 2 or 5}$$

P in 3 or 6

$$\left.\begin{array}{l}\text{A — 6, 2, 5 or 3}\\ \text{B — 6, 7, and 3}\\ \text{C — 6, 4, 1 or 3}\end{array}\right\} P' \text{ in 6 or 3}$$

P in 7

$$\left.\begin{array}{l}\text{A — 7, 1, 4}\\ \text{B — 7, 2, 5}\\ \text{C — 7, 3, 6}\end{array}\right\} P' \text{ in 7}$$

In the same manner we locate P' for P in any region when the involution is hyperbolic at C while elliptic at A and B, or hyperbolic at A while elliptic at B and C, or hyperbolic at B while elliptic at C and A. The complete results may be tabulated as follows:

Involution is hyperbolic at			*A B C*	*C*	*A*	*B*
P lies in	1 or 4	P' lies in	4 or 1	6 or 3	7	2 or 5
	2 or 7		2 or 5	7	6 or 3	4 or 1
	3 or 6		6 or 3	4 or 1	2 or 5	7
	7		7	2 or 5	4 or 1	6 or 3

This investigation shows that the regions 1 and 4 are interchangeable; likewise, 2 and 5, 3 and 6. Indeed, we see that one can pass from 1 to 4 without crossing a side of the triangle; similarly, from 2 to 5, and from 3 to 6. We may, therefore, regard 1 and 4 as one region, R_1; 2 and 5 as one region, R_2; and 3

and 6 as one region, R_3. 7 stands by itself as R_4. The above table may then be written:

Involution is hyperbolic at			$A\ B\ C$	C	A	B
P lies in	R_1	P' lies in	R_1	R_3	R_4	R_2
	R_2		R_2	R_4	R_3	R_1
	R_3		R_3	R_1	R_2	R_4
	R_4		R_4	R_2	R_1	R_3

The transformation of P indicated by the first column leaves R_1, R_2, R_3, R_4 all unaltered; that is, it is effected by the substitution (1) (2) (3) (4). The transformation of the second column is effected by the substitution (1, 3) (2, 4); that of the third column by the substitution (1, 4) (2, 3); and that of the fourth by the substitution (1, 2) (3, 4). These four substitutions

(1) (2) (3) (4)
(1, 3) (2, 4)
(1, 4) (2, 3)
(1, 2) (3, 4)

constitute the well known "Group of Four."

Any given curve C intersects the sides of the triangle in a definite order, thus establishing the order in which the corresponding curve C' must pass through the vertices. This enables us to determine for a given branch of the curve C how the corresponding branch of the curve C' must pass through the region in which it lies. We can, therefore, for a given curve C determine at once the form of the corresponding curve C'. Consider, for example, the quartic that will correspond to the conic C in figure 2, where the involution is hyperbolic at every vertex. Reading from right to left, C intersects the sides of the triangle in the order $a\ b\ c\ a\ b\ c$. Denote the portions of C which lie in 1, 2, 3, 4, 5, 6, by c_1, c_2, c_3, c_4, c_5, and c_6, respectively, and the corresponding portions of C', c'_1, c'_2, c'_3, etc.,

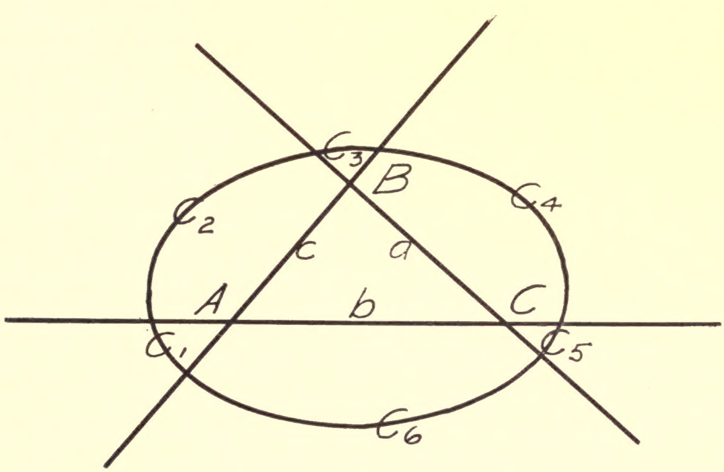

Fig. 2

respectively. c_1 lies in 1 and between b and c. c'_1, then, must lie in 4 or 1 and between B and C. Moving continuously along C' we may pass from B to C directly so that c'_1 lies entirely in 4, as in figure 1, or indirectly by way of infinity so that c'_1 lies partly in 4 and partly in 1, as in figure 3. In the same way we determine c'_2, c'_3, c'_4, c'_5, and c'_6, and, excluding the possibility of infinite branches of the kind shown in figure 3, we find that C' must for the conic C in figure 2 have the form given in figure 6.

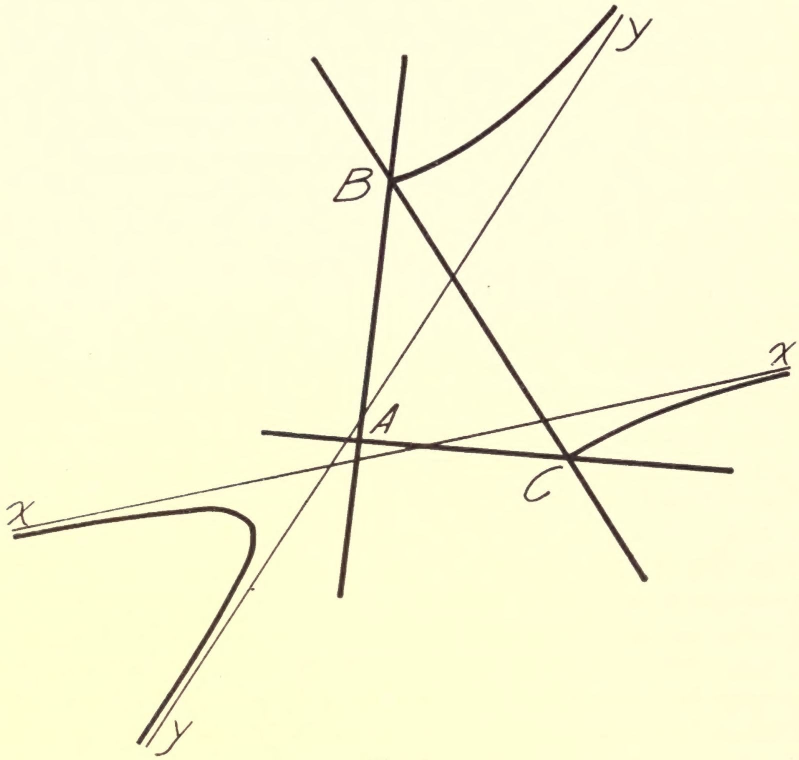

Fig. 3

11. *Classification of Unicursal Trinodal Quartics.*—If the given curve C is a conic or a curve of degree greater than the second, the position of the curve C with reference to the triangle affords a basis for classification of the curves C', since the order in which the curve C intersects the sides of the triangle is the order in which C' must pass through the vertices. The unicursal quartic is of particular interest here.

A conic section according to its relative position may intersect the sides of the triangle in five essentially different orders. These are:

(1) $a\ a\ b\ b\ c\ c$
(2) $a\ b\ c\ a\ b\ c$
(3) $a\ b\ a\ b\ c\ c$
(4) $a\ a\ c\ b\ b\ c$
(5) $a\ c\ a\ b\ c\ b$

We may describe the position of the conic by noting the successive regions through which it passes. Thus the position of the conic in figure 2 is given by noting that it passes successively through the regions 1, 2, 3, 4, 5, 6.

The same order may be given by more than one position of the conic. Order (1) is given by four positions, namely,

4 7 6 7 2 7,
3 2 1 2 7 2,
7 4 5 4 3 4,
5 6 7 6 1 6;

order (2) by one position,

6 5 4 3 2 1;

order (3) by three positions,

4 7 6 5 4 3,
6 5 4 7 6 1,
7 4 5 6 7 2;

order (4) by four,

3 2 3 4 5 4,
6 5 6 1 2 1,
2 3 2 7 6 7,
7 4 7 2 1 2;

and order (5) by one position,

2 3 4 7 6 1.

Any particular order represents more than one class of unicursal quartics, since each of the four possible forms of the involution may give rise to a different kind of curve for a given position of the conic. An analysis of all cases shows that a change of the conic from one position to another representing the same order is equivalent to a possible change in the character of the involution. It follows that all the classes represented by one order may be obtained from any one position of the conic which produces that order by changing the character of the involution.

It is not difficult by the method indicated above to ascertain for the conic in any given position and under any form of the involution the number of classes represented by each order and the form of the curve (excluding the possibility of infinite branches such as those shown in figure 3). The complete results may be conveniently arranged in the following table:

Order	No. of Classes	Position	Involution Hyperbolic at	Form of Quartic
a a b b c c	2	476727	*A*, *B* and *C*	Fig. 4
			C, or *A*, or *B*	Fig. 5
		321272	*C*	Fig. 4
			A, or *B*, or *A*, *B* and *C*	Fig. 5
		745434	*A*	Fig. 4
			B, or *C*, or *A*, *B* and *C*	Fig. 5
		567616	*B*	Fig. 4
			C, or *A*, or *A*, *B* and *C*	Fig. 5
a b c a b c	2	654321	*A*, *B* and *C*	Fig. 6
			A, or *B*, or *C*	Fig. 7
a b a b c c	3	745672	*A*, *B* and *C*	Fig. 8
			C	Fig. 9
			A, or *B*	Fig. 10
		476543	*A*	Fig. 8
			B	Fig. 9
			C, or *A*, *B* and *C*	Fig. 10
		654761	*B*	Fig. 8
			A	Fig. 9
			C, or *A*, *B* and *C*	Fig. 10
a a c b b c	3	323454	*A*, *B* and *C*	Fig. 11
			C	Fig. 12
			A, or *B*	Fig. 13
		656121	*A*, *B* and *C*	Fig. 11
			C	Fig. 12
			A, or *B*	Fig. 13
		232767	*A*	Fig. 11
			B	Fig. 12
			C, or *A*, *B* and *C*	Fig. 13
		747212	*B*	Fig. 11
			A	Fig. 12
			C. or *A*, *B* and *C*	Fig. 13
a c a b c b	2	234761	*A*, *B* and *C*, or *C*	Fig. 14
			A, or *B*	Fig. 15

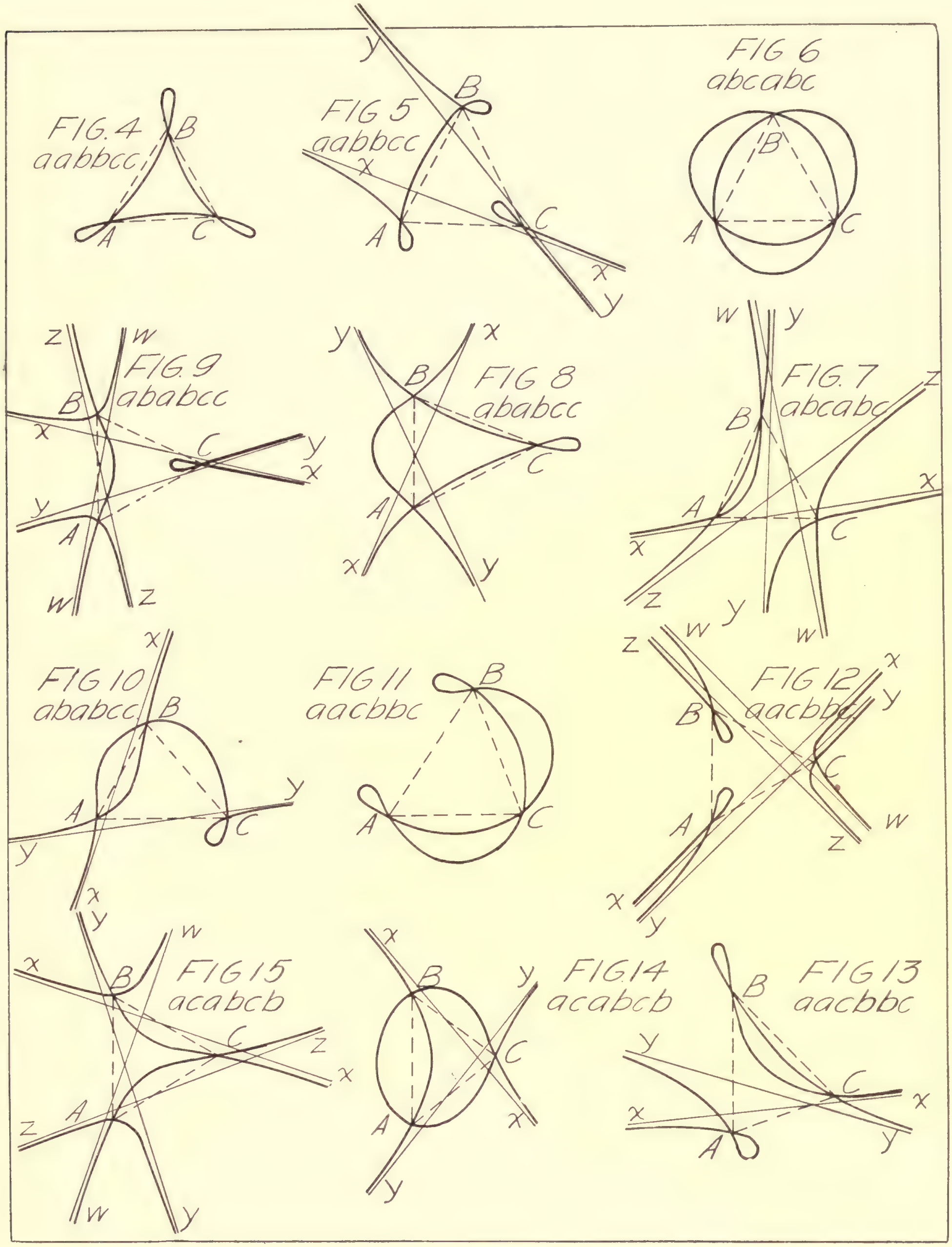
FIG. 4
aabbcc
FIG 5
aabbcc
FIG 6
abcabc
FIG. 9
ababcc
FIG 8
ababcc
FIG. 7
abcabc
FIG 10
ababcc
FIG 11
aacbbc
FIG 12
aacbbc
FIG 15
acabcb
FIG. 14
acabcb
FIG 13
aacbbc

Figs. 4–15

In all, twelve classes of unicursal trinodal quartics are obtained in this way. Further distinction may be made, according as the intersections of the conic with the sides of the triangle are real and distinct, real and coincident, or conjugate-imaginary.

Special forms of the self-polar triangle give rise, of course, to special kinds of curves.

Chapter II

THE UNICURSAL QUARTIC AS THE LOCUS OF THE INTERSECTION OF CORRESPONDING RAYS OF TWO PROJECTIVE PENCILS OF THE SECOND ORDER

1. The curve is defined as the locus of the points of intersection of corresponding rays of two projective pencils of the second order.

2. *Theorem.—The locus described is a unicursal curve having, in general, three nodes, and intersecting any line in the plane in at most four points.*

See Chapter I, §§ 3 and 6.

3. *Notations.*—One pencil of the second order (and also the conic enveloped by it) will be denoted by κ; the other by κ'. The rays of each pencil will be denoted by α, β, γ, ... and α', β', γ', . . . respectively. The tangents to κ and κ' from the points of the quartic and other than α, β, γ, ... and α', β', γ', ... will be denoted by a, b, c, ... and a', b', c', ... respectively. The quartic itself will be denoted by Q.

4. *Theorem.—No point of Q lies within κ or κ'.*

This is obvious from the definition of the curve given in § 1.*

5. *Theorem.—The quartic Q touches each conic κ and κ' in at most four points.*

Take a point S on κ for the center of a pencil of the first order perspective to κ. This generates with κ' a cubic with a double point or node at S.† The cubic can intersect κ in at most four other points besides S. These are easily seen to be points of the locus Q.

6. *Problem.*—Given κ and κ' with three pairs of corresponding rays, to construct Q. (See figure 16.)

Let the three pairs of rays be α, α'; β, β'; and γ, γ'. The intersection of α and α' will be a point A on Q. Likewise β and β' determine B on Q and γ and γ', C on Q. Draw through A to κ the other tangent a; also to κ' the other tangent a'. Take a perspective to κ and a' perspective to κ'. a and a' are two point-rows in perspective position since they have a self-corresponding point, A. Their center of perspectivity may be found by joining any two pairs of corresponding points on a and a', such as (a, β) with (a', β') and (a, γ) with (a', γ'). Denote this center of perspectivity by Σ_A. Then to find the point X of Q determined by any ray ξ of κ, we first join the point (ξ, a) with Σ_A. This line will intersect a' in the point (ξ, a'). The intersection of ξ and ξ' is the desired point X.

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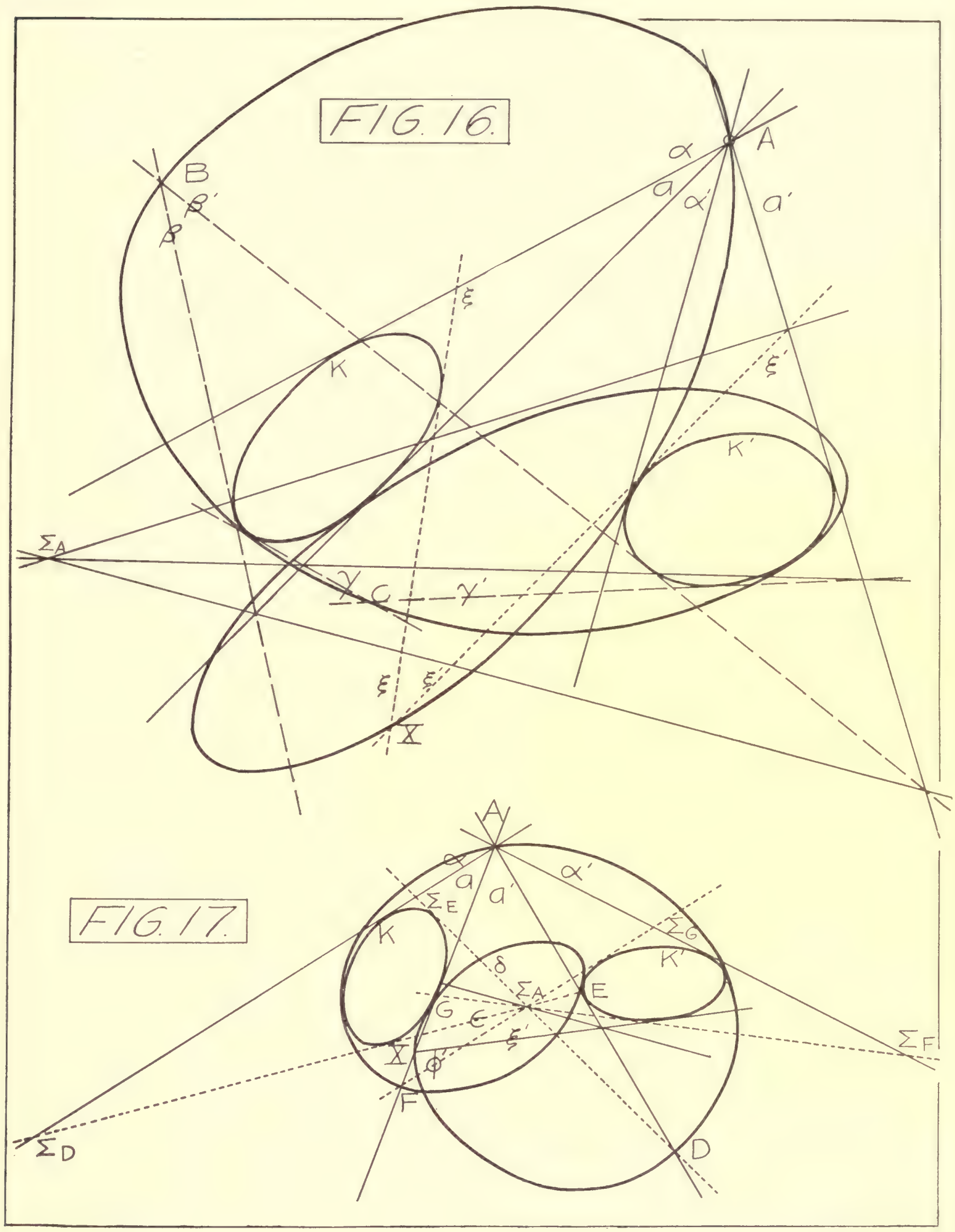
* Unless otherwise stated, the sections referred to are those of Chapter II.

† See Drasch, "Beitrag zur synthetischen Theorie der ebenen Curven dritter Ordnung mit Doppelpunkt," *Wiener Berichte*, vol. 85 (1882), p. 534. Also see D. N. Lehmer, "Constructive Theory of the Unicursal Cubic by Synthetic Methods," *Trans. Am. Math. Soc.*, vol. 3, p. 372.



Figs. 16 and 17

7. *Theorem.—Every point P of the quartic Q has its corresponding point Σ_P.* The locus of all such points will be denoted by Σ.

8. *Problem.*—Given κ and κ' with one pair of corresponding rays and the corresponding point of Σ, to construct Q.

Let the given pair of corresponding rays be α, α'. The intersection of α and α' will be the point A of Q. As in § 6, draw a and a', taking them perspective to κ and κ', respectively. Using Σ_A, which is given, as the center of perspectivity, the ray of κ' corresponding to any ray of κ, or vice versa, may at once be found, as before.

9. *Theorem.—The tangents from Σ_A to κ meet a' in points of Q. Likewise, the tangents from Σ_A to κ' meet a in points of Q.* (See figure 17.)

This is seen to be true by drawing the rays of κ' which correspond to the tangents from Σ_A to κ; similarly, those of κ which correspond to the tangents from Σ_A to κ'.

10. *Problem.*—To find the fourth point of Q upon a or a'. (It is the point X in figure 17.)

To do this we consider a as a ray ξ of κ. ξ meets a in the point of contact of a and κ. Join this point with Σ_A. The line so obtained intersects a' in the point (ξ', a'). The intersection of ξ' with ξ (or a) is the desired point.

Instead of the point-rows a and a' and their center of perspectivity, Σ_A, we may make use of any other such set, as b, b' and Σ_B. In this case a, considered as a ray ξ of κ, appears as an ordinary ray and ξ' is found in the usual manner indicated in § 6.

The fourth point of Q upon a' is found in precisely the same manner as the fourth point of Q upon a.

11. We are now in a position to state the following:

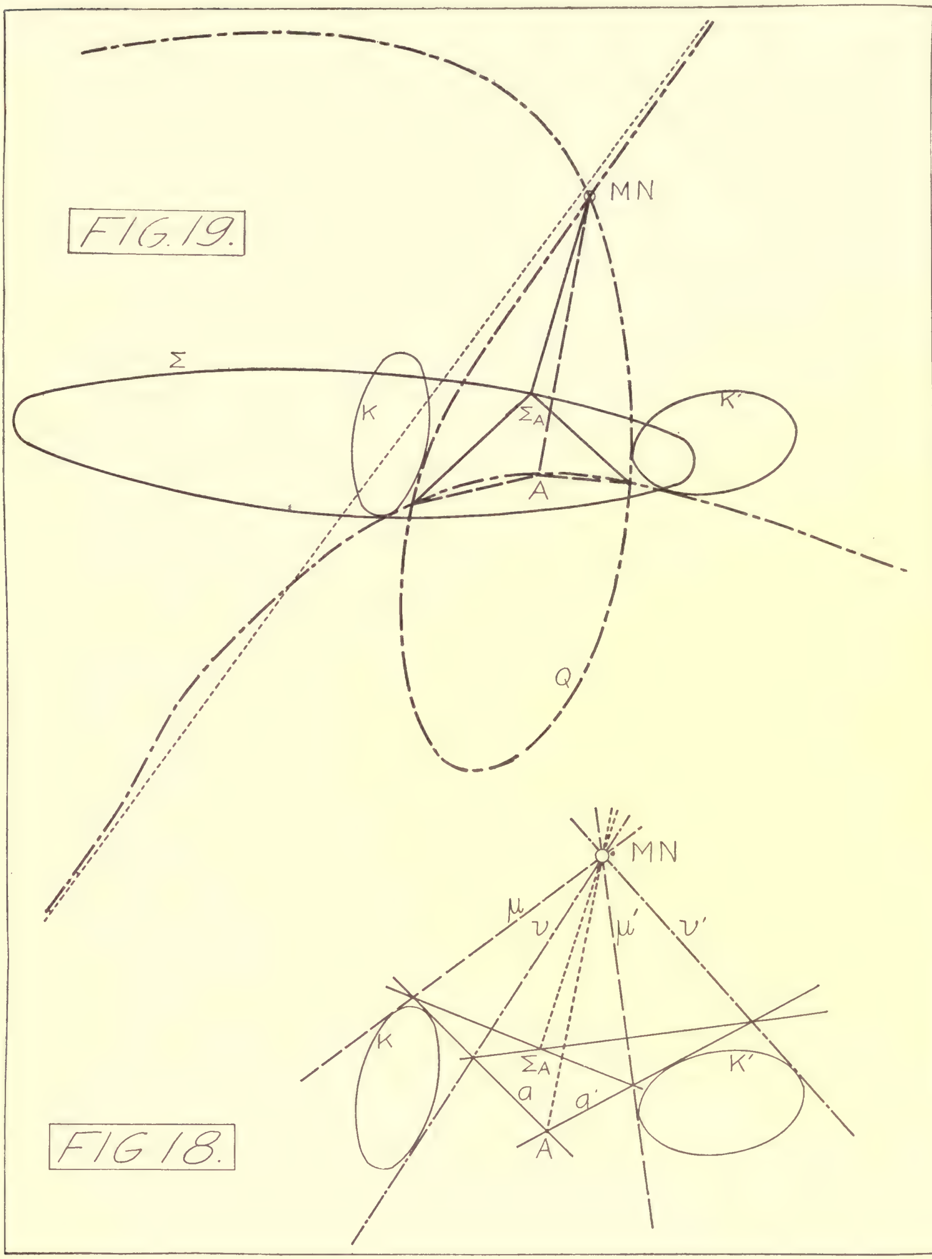
Theorem.—The tangents from Σ_A to κ meet a' in points of Q and α in points of Σ. Likewise, the tangents from Σ_A to κ' meet a in points of Q and α' in points of Σ. The point of Σ on α (or α') determined by one tangent from Σ_A to κ (or κ') corresponds to the point of Q on a' (or a) in which a' (or a) is met by the other tangent from Σ_A to κ (or κ'). (See figure 17.)

The first part of the theorem was proved in § 9.

Denote one tangent to κ from Σ_A by δ and the other one by ϵ. Then it follows from § 9 that (δ, a') is D and (ϵ, a') is E. d' and e' coincide with a'. In finding Σ_D, draw the line $(\alpha, d — \alpha' d')$. This is α itself. Therefore, Σ_D lies on α. Similarly, we show that Σ_E lies on α.

Instead of Σ_A start with Σ_D, drawing the tangents from Σ_D to κ. One of these is α. The other must meet d', which is a', in a point of Q. Obviously, it cannot meet d' in A or D. Nor can it meet d' in the point X found in § 10, since for X, a' is ξ' and not x'. It must then meet d' in E and, therefore, the second tangent from Σ_D to κ is ϵ. But ϵ passes through Σ_A. That is, Σ_A and Σ_D lie on ϵ. In like manner, starting with Σ_E, it may be shown that Σ_E and Σ_A lie on δ. Thus one tangent δ from Σ_A to κ meets a' in a point D of Q and α in the point Σ_E, while the other tangent ϵ from Σ_A to κ meets a' in the point E of Q and α in Σ_D.

In precisely the same manner the theorem is proved for the tangents from Σ_A to κ' instead of κ.



Figs. 18 and 19

12. *Problem.*—Given κ and κ' with one pair of corresponding rays, and the corresponding point of Σ, to find the point of Q corresponding to any particular point of Σ.

Let the given pair of rays be $\beta\ \beta'$, and the corresponding point of Σ, Σ_B. Since β and β' are known, b and b' are known. Let the particular point of Σ be Σ_A. To find A, draw the tangents from Σ_A to κ and κ', calling them δ, ϵ, θ', and ι'. Using Σ_B as a center of perspectivity of the point rows b and b', construct δ', ϵ', θ, and ι, thus obtaining D, E, H, and I. D and E determine a' and H and I determine a. (a, a') is A.

13. *Theorem.—The locus of the points Σ, the centers of perspectivity corresponding to the points of Q, is a conic section from which Q is generated by means of a quadratic reciprocal transformation.* (See figures 18 and 19.)

A quartic generated from a conic by means of the quadratic reciprocal transformation discussed in Chapter I had its nodes at the vertices of the singular triangle, the involution centers.

Consider the four tangents from a node of Q, two to κ and two to κ'. (See figure 18.) They are two pairs of corresponding rays of the pencils κ and κ'. Call them μ, μ', and ν, ν'. One pair determines the node as M, considered as lying on one branch of the curve, the other determines it as N, considered as lying on the other branch. Using μ, μ', and ν, ν', find Σ_A for A. This is done by marking the intersection of the line $(a, \mu - a', \mu')$ with the line $(a, \nu - a', \nu')$. The points

$$A, \Sigma_A;\ (a, \mu),\ (a', \nu');\ (a, \nu),\ (a', \mu')$$

are three pairs of opposite vertices of a complete quadrilateral and therefore project to any point of the plane in an involution of rays. They project to the node MN in three pairs of rays, two of which are μ, ν', and μ', ν. But these lines are fixed and do not depend upon A or Σ_A.

Now consider the involutions at any two of the three nodes of Q. The lines from any given point Σ_A of Σ to the nodes correspond in involution to rays which intersect in the point A of Q corresponding to Σ_A. As Σ_A moves on Σ, A describes the quartic Q. Thus Σ is exhibited as that curve from which the quartic is generated by the quadratic reciprocal transformation and is, therefore, a conic section. (Chapter I, §§ 3 and 8.)

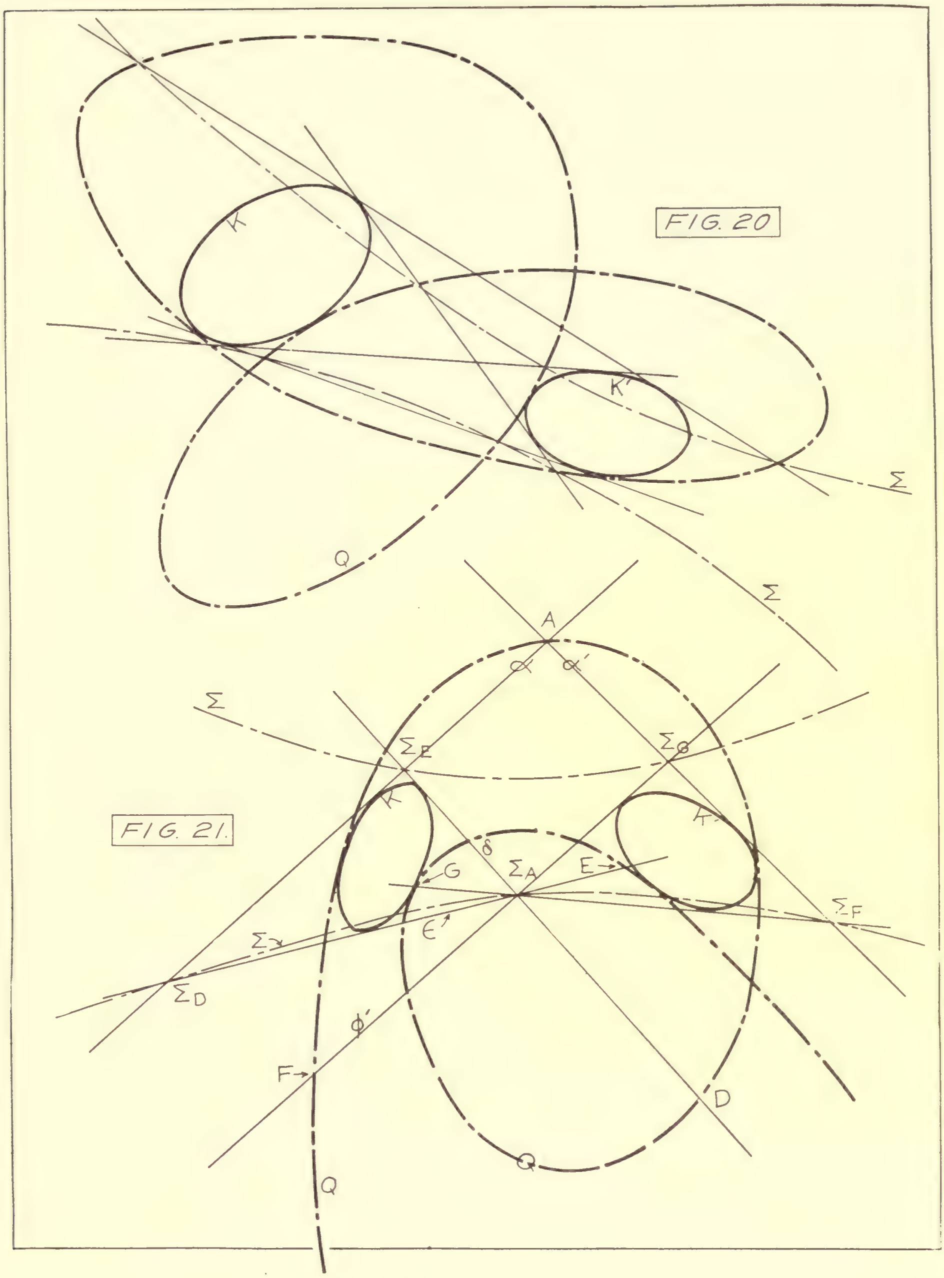
14. *Theorem.—The points of intersection of Q and Σ lie on the tangents common to κ and κ'.* (See figure 20.)

Let a common tangent of κ and κ' be a ray β in the pencil κ. It corresponds to a ray β' of κ', in general distinct from β. b' then coincides with β. The tangents from Σ_B to κ meet β in points of Σ and b' in points of Q. But β and b' coincide.

15. *Theorem.—κ (or κ') and Σ are two conics such that a triangle circumscribed about κ (or κ') is inscribed in Σ.*

This follows from § 11, the tangents there denoted α, δ, and ϵ forming such a triangle, of which the vertices are Σ_E, Σ_A, and Σ_D. (See figures 17 and 21, triangles $\Sigma_A\ \Sigma_D\ \Sigma_E$ and $\Sigma_A\ \Sigma_F\ \Sigma_G$.)

For the invariant relation connecting two conics related as in this theorem see Salmon, *Conic Sections,* § 376.



Figs. 20 and 21

16. *Theorem.—When Σ_A lies on κ, a' is tangent to Q and a is tangent to Σ; similarly, when Σ_A lies on κ', a is tangent to Q and a' is tangent to Σ.* (See figure 22.)

In this case the two tangents from Σ_A to κ, δ and ϵ, coincide. Consequently, a' meets Q in two coincident points, D and E, and a meets Σ in two coincident points Σ_E and Σ_D. Similarly, for Σ_A on κ'.

Differently stated, the latter part of this theorem tells us:

The tangents common to κ (or κ') and Σ are obtained by drawing the tangents to κ (or κ') from those points of Σ where the tangents to κ (or κ') at the intersections of κ (or κ') and Σ again intersect Σ.

Thus κ (or κ') and Σ have as many common tangents as intersections.

17. *Theorem.—If κ (or κ') and Σ do not intersect κ (or κ') lies wholly within Σ.* (See figures 23 and 24.)

If κ (or κ') and Σ do not intersect, either Σ lies wholly within κ (or κ') or κ (or κ') lies wholly within Σ. Otherwise common tangents could be drawn. But Σ cannot lie wholly within κ (or κ') since a triangle circumscribed about κ (or κ') is inscribed in Σ. Therefore, if κ (or κ') and Σ do not intersect, κ (or κ') lies wholly within Σ.

18.—*Theorem.—a and a', a' and a determine an involution of rays at A in which $A\,\Sigma_A$ corresponds to the tangent to Q at A.* (See figures 24 and 25.)

Let D and E be the points of Q on a determined by the tangents from Σ_A to κ. κ' and F and G the points of Q on a' determined by the tangents from Σ_A to κ. Consider, then, the three pairs of points A and Σ_A, E and Σ_E, F and Σ_F. They are obviously not the three pairs of opposite vertices of a complete quadrilateral, since A, Σ_A, E, and Σ_E determine a quadrilateral of which D and Σ_D are the third pair of opposite vertices; and A, Σ_A, F and Σ_F determine a quadrilateral of which G and Σ_G are the third pair of opposite vertices. The locus of points from which the three given pairs of points are seen in involution is a general plane cubic C, passing through all the six given points and through D, Σ_D, G and Σ_G.* From § 13 we know that the nodes of Q also satisfy the condition for points of C.

The cubic C is tangent to Q at A and to Σ at Σ_A.

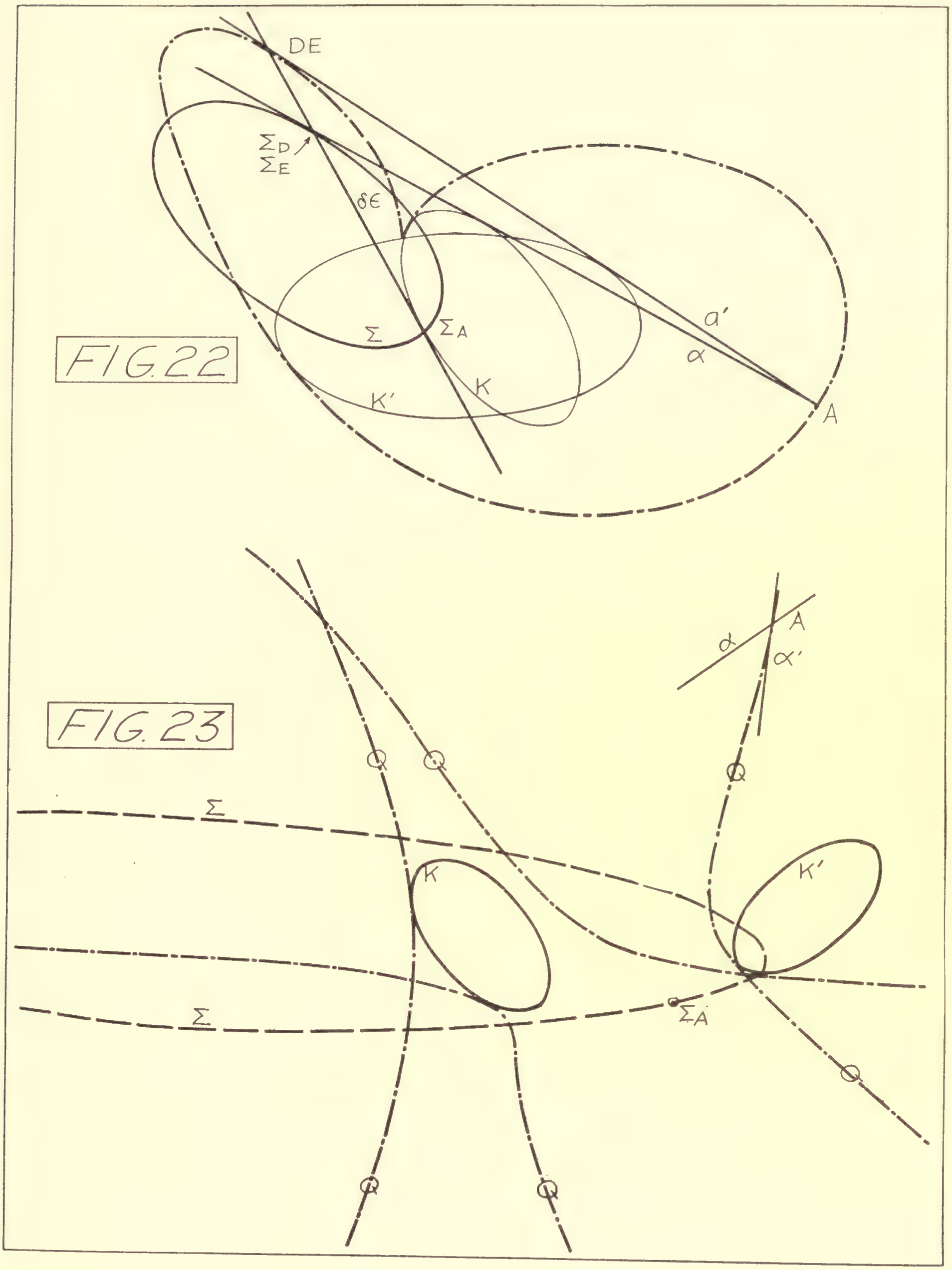
In the quadratic reciprocal transformation a point A of Q goes into Σ_A of Σ. The tangent line to Q at A goes into a conic through the three involution centers, the nodes of Q (see Chapter I, § 6), and tangent to Σ at Σ_A. A point of the cubic C passing through the three involution centers goes into the conjugate point of C.† A and Σ_A are conjugate points of the cubic C. The tangent line to C at A corresponds to a conic through the three involution centers and tangent to C at Σ_A. If the tangent lines to C and to Q at A coincide, the corresponding conics must coincide; that is, if C is tangent to Q at A it must be tangent to Σ at Σ_A.

The cubic C is tangent to Σ at Σ_A.

Construct the tangent to Σ at Σ_A by means of Pascal's theorem, using the points Σ_A, Σ_D, Σ_E, Σ_F, Σ_G, and numbering them 16, 2, 3, 4, 5, respectively. (See figure 25.)

* See Schroeter, *Ebene Curven Dritter Ordnung*, Leipzig, 1888; § 1, 6; § 2, 1–4.

† See D. N. Lehmer, "On the Combination of Involutions," *Am. Math. Mo.*, vol. 18, no. 3 (March, 1911).



Figs. 22 and 23

The tangent line to the cubic C at Σ_A is that ray which corresponds to the ray $\Sigma_A\, A$ in the involution determined at Σ_A by two pairs of conjugate rays, as:

$$(\Sigma_A\, E) \text{ and } (\Sigma_A\, \Sigma_E)\,;\ (\Sigma_A\, F) \text{ and } (\Sigma_A\, \Sigma_F).^*$$

To obtain this tangent line to C at Σ_A, construct a complete quadrilateral, two of whose pairs of opposite vertices will project to Σ_A in $(\Sigma_A\, E)$ and $(\Sigma_A\, \Sigma_E)$; $(\Sigma_A\, F)$ and $(\Sigma_A\, \Sigma_F)$; and of the third pair, one vertex in $(\Sigma_A\, A)$. (See figure 25.) The other vertex must lie on the ray $(\Sigma_A\, \Sigma_A)$, the tangent to C at Σ_A. Such a complete quadrilateral is determined by the four lines: α, α', $(\Sigma_E\, \Sigma_F)$, and the Pascal line found in constructing the tangent to Σ at Σ_A. But the vertex which projects to Σ_A giving the tangent to C at Σ_A is the intersection of the Pascal line and the line (3, 4) in the construction of the tangent to Σ at Σ_A; that is to say, the point which with Σ_A gave (6, 1), the tangent to Σ at Σ_A. The tangent to Σ at Σ_A and the tangent to C at Σ_A are, therefore, one and the same line; that is, the cubic C is tangent to Σ at Σ_A. It is, therefore, tangent to Q at A.

Hence, to construct the tangent to Q at A, we need only to construct the tangent to C at A. This is done, as in the case of the tangent to C at Σ_A, by constructing the ray which corresponds to the ray $A\, \Sigma_A$ in the involution determined at A by two pairs of conjugate rays, as:

$$(A\, E) = \alpha, \text{ and } (A\, \Sigma_E) = \alpha'\,;$$
$$(A\, F) = \alpha' \text{ and } (A\, \Sigma_F) = \alpha.$$

This may be done in the following manner. (See figures 24 and 25.) On the line $A\, \Sigma_A$ select any point other than A. Through it draw two lines l and l'. Mark the intersections of l with α and α' and those of l' with α and α'. The intersection of the lines $(l\alpha - l'\alpha)$ and $(l\alpha' - l'\alpha')$ must lie on the tangent to Q at A.

19. *The Cubic C.*—For every point of Q there is a cubic C such as that described in § 18. It may be denoted by C_A, C_B, etc., the subscript indicating the point of Q at which the cubic is tangent.

We observe that we have at once all the intersections of C_A, both with the quartic Q and the conic Σ. The three nodes of Q account for six intersections of Q and C_A, the tangency at A for two more, and the remaining four are at the points denoted by D, E, F, and G in § 18. C_A is tangent to Σ at Σ_A and intersects it in the four remaining points Σ_D, Σ_E, Σ_F, and Σ_G.

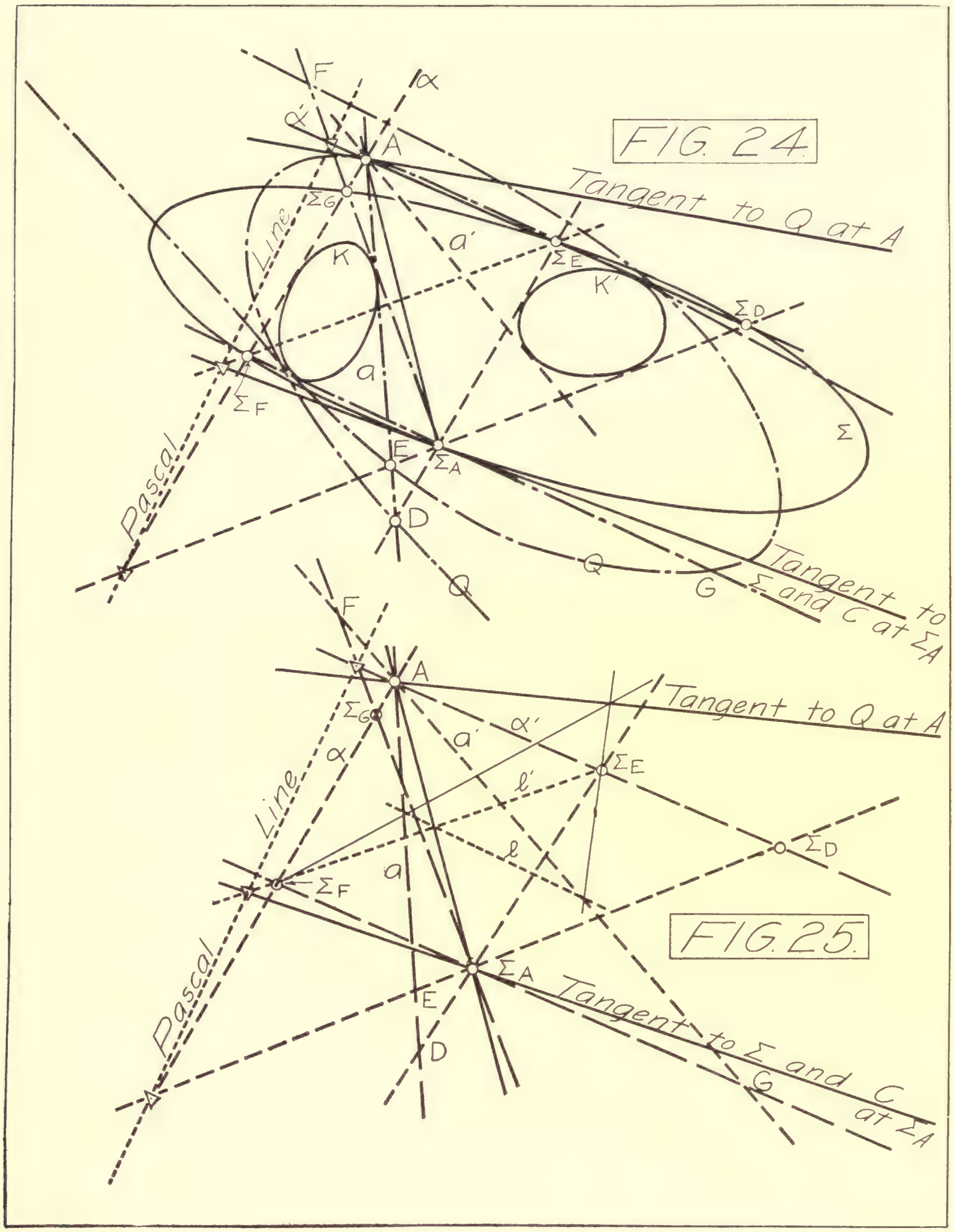
Incidentally, the existence of the cubics C gives a method of determining the nodes of Q since through them all the cubics C must pass.

20. *Theorem.—The class of Q is 6.*

A tangent line to Σ from a node of Q, that is, from an involution center S_1, corresponds to a tangent to Q from S_1. Since from any point in the plane only two tangents may be drawn to a conic section, only two tangents may be drawn from S_1 to Σ and, hence, only two from S_1 to Q. There are two tangents to Q at S_1, each of which counts as two tangents to Q from a point of the plane not on Q. In all, then, at most four tangents may be drawn to Q from a node S_1, or at most six from an ordinary point of the plane.

Consistent with this is the number of tangents common to κ (or κ') and Q.

* See Schroeter, *op. cit.*; § 2, 6.



Figs. 24 and 25

From § 16 we know that every intersection of κ' (or κ) and Σ yields a tangent common to κ (or κ') and Q. Since Σ and κ' (or κ) can intersect in at most four points, this gives rise to only four tangents common to κ (or κ') and Q. The remaining eight lie at the four points where Q may touch κ (or κ'). (See § 5.)

21. *Theorem.—The six tangents drawn from the three nodes to the quartic are tangent to one and the same conic.* (See figure 26, conic I.)

The lines in question correspond by involution to the six tangents from the three nodes to the Σ conic. Denote the nodes by S_1, S_2, and S_3; the tangents from them to Σ by 1, 2; 3, 4; and 5, 6, respectively; and the lines corresponding to 1, 2, 3, 4, 5, 6, by 1′, 2′, 3′, 4′, 5′, 6′, respectively. Since the lines 1, 2, 3, 4, 5, 6 are tangent to a conic, the lines

$$(1, 2 — 4, 5),\ (2, 3 — 5, 6),\ (3, 4 — 6, 1)$$

are concurrent. (Brianchon.) But these lines are

$$(S_1 — 4, 5),\ (S_2 — 5, 6),\ (S_3 — 6, 1).$$

To them correspond

$$(S_1 — 4', 5'),\ (S_2 — 5', 6'),\ (S_3 — 6', 1'),$$

respectively. Since the first three are concurrent the second three must be. But the second three are

$$(1', 2' — 4', 5'),\ (2', 3' — 5', 6'),\ (3', 4' — 6', 1'),$$

and therefore 1′, 2′, 3′, 4′, 5′, 6′ are tangent to one and the same conic section. (Brianchon.)

22. *Theorem.—The tangents to Q at a node correspond in involution to the lines joining the node with those points of Σ which lie on the line joining the other two nodes.*

For from Chapter I, § 5, it follows that Σ_M and Σ_N, corresponding to the node $M\ N$, lie on the side of the singular triangle opposite to $M\ N$, that is, on the line joining the other two nodes. $M\ \Sigma_M$ corresponds to $M\ M$, the tangent to Q at M, and $N\ \Sigma_N$ corresponds to $N\ N$, the tangent to Q at N.

23. *Theorem.—The six tangent lines to Q at the three nodes are tangent to a conic section.* (See figure 26, conic II.)

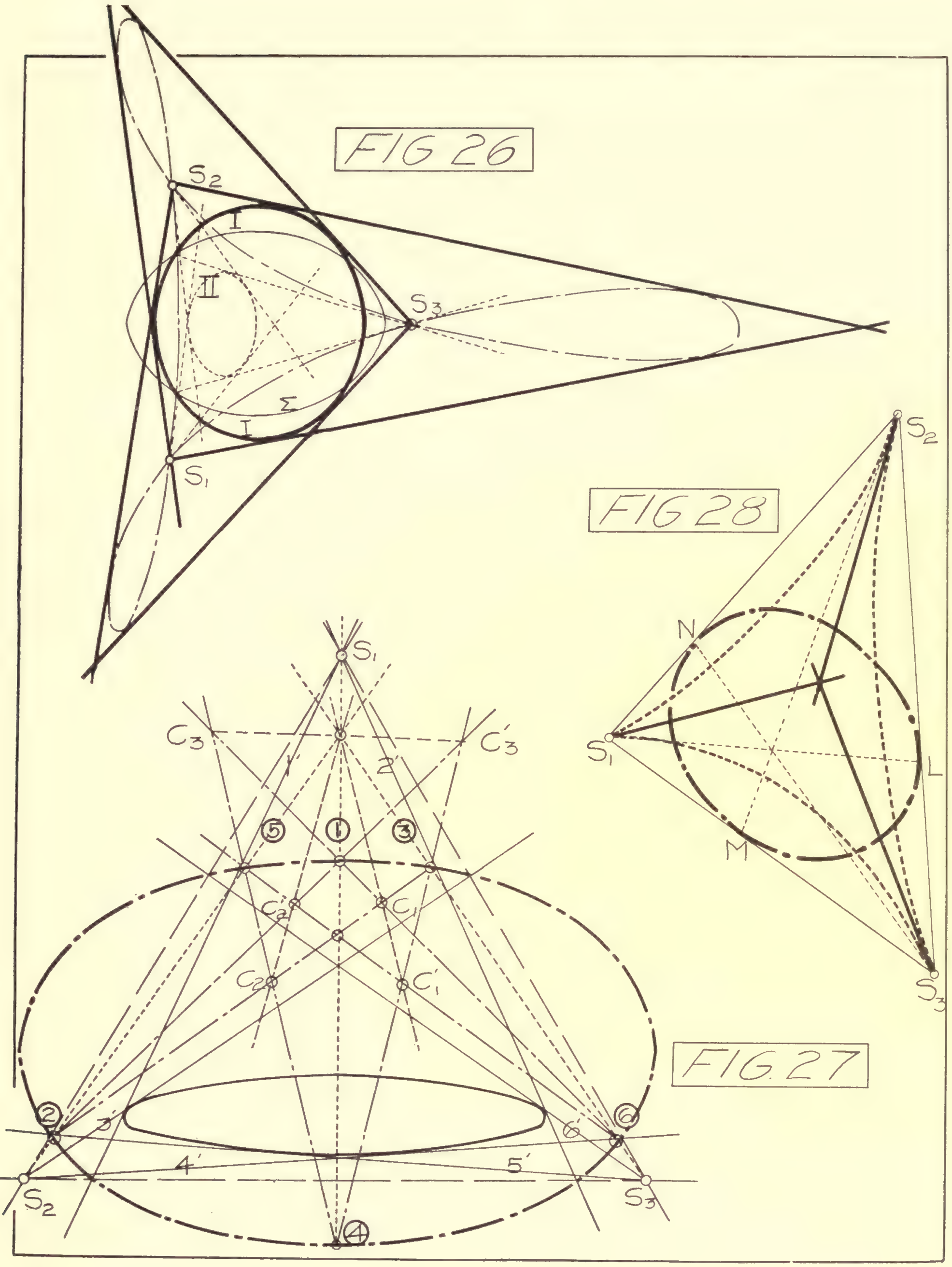
As before in § 21, denote the nodes of Q by S_1, S_2, and S_3. From Brianchon's theorem we know that if the six lines in question circumscribe a conic section the three lines joining the opposite vertices of the hexagon they from all pass through one and the same point. Number the two tangents to Q at S_1, 1 and 2; those at S_2, 3 and 4; and those at S_3, 5 and 6. The lines

$$(1, 2 — 4, 5),\ (2, 3 — 5, 6),\ (3, 4 — 6,1)$$

join opposite vertices of the hexagon.

In the involutions at S_1, S_2, and S_3 there correspond to 1, 2, 3, 4, 5, 6, lines which we shall call 1′, 2′, 3′, 4′, 5′, and 6′, respectively.

If the lines 1, 2, 3, 4, 5, 6 are tangent to a conic section, the lines 1′, 2′, 3′, 4′, 5′, 6′ are also tangent to a conic section, and vice versa. For the point (4, 5) goes into the point (4′, 5′) and the line



Figs. 26–28

$$(S_1 - 4, 5) \text{ corresponds to } (S_1 - 4', 5').$$

Similarly,

$$(S_2 - 6, 1) \text{ corresponds to } (S_2 - 6', 1')$$

and

$$(S_3 - 2, 3) \qquad \text{to} \qquad (S_3 - 2', 3').$$

The intersection of any two of the three,

$$(S_1 - 4, 5),\ (S_2 - 6, 1),\ (S_3 - 2, 3),$$
$$\text{as } (S_1 - 4, 5) \text{ and } (S_2 - 6, 1),$$

corresponds to the intersection of the corresponding two, as

$$(S_1 - 4', 5') \text{ and } (S_2 - 6', 1').$$

It follows that if the three lines

$$(S_1 - 4', 5'),\ (S_2 - 6', 1'),\ (S_3 - 2', 3'),$$

that is, the lines

$$(1', 2' - 4', 5'),\ (3', 4' - 6', 1'),\ (5', 6' - 2', 3'),$$

are concurrent, the three corresponding lines

$$(S_1 - 4, 5),\ (S_2 - 6, 1),\ (S_3 - 2, 3),$$

that is, the lines

$$(1, 2 - 4, 5),\ (3, 4 - 6, 1),\ (5, 6 - 2, 3)$$

are also concurrent, and vice versa. If the lines

$$(1', 2' - 4', 5'),\ (3', 4' - 6', 1'),\ (5', 6' - 2', 3')$$

are concurrent, the lines $1', 2', 3', 4', 5', 6'$ are tangent to a conic section. In that case, the lines 1, 2, 3, 4, 5, 6 are also tangent to a conic section.

$1'$, $2'$, $3'$, $4'$, $5'$, $6'$ are lines joining the vertices of a triangle, each with the points in which the opposite side intersects a given conic section. (See § 22.) *Any six such lines circumbscribe a conic section.* (See figure 27.)

Mark the six intersections of the three lines

$$(1', 2', - 4', 5'),\ (2', 3' - 5', 6'),\ (3', 4' - 6', 1')$$

with the given conic, numbering them in any order (1), (2), (3), (4), (5), (6); say, so that (1) and (4) lie on the line $(1', 2' - 4', 5')$, (2) and (5) on the line $(3', 4' - 6', 1')$, and (3) and (6) on the line $(2', 3' - 5', 6')$. These six points form an inscribed hexagon whose opposite sides must, therefore, intersect in three collinear points. (Pascal.) The three collinear points are:

$$\{(1)\ (2) - (4)\ (5)\},\ \{(2)\ (3) - (5)\ (6)\},\ \{(3)\ (4) - (6)\ (1)\}.$$

Now make use of the following notation:*

$$\begin{array}{lll} (1) = A_1 & (2) = B_1 & \{(2)(3) - (6)(1)\} = C_1 \\ (4) = A'_1 & (5) = B'_1 & \{(5)(6) - (3)(4)\} = C'_1 \\ (3) = A_2 & (4) = B_2 & \{(2)(3) - (4)(5)\} = C_2 \\ (6) = A'_2 & (1) = B'_2 & \{(5)(6) - (1)(2)\} = C'_2 \\ (5) = A_3 & (6) = B_3 & \{(4)(5) - (6)(1)\} = C_3 \\ (2) = A'_3 & (3) = B'_3 & \{(1)(2) - (3)(4)\} = C'_3 \end{array}$$

* Care must be taken to have corresponding vertices lie on the lines $(1', 2' - 4', 5')$, $(2', 3' - 5', 6')$, and $(3', 4' - 6', 1')$.

$A_1 B_1 C_1$ and $A'_1 B'_1 C'_1$ are two triangles so situated that their corresponding sides intersect in three collinear points. Therefore, the lines joining corresponding vertices, that is, the lines

$$A_1 A'_1 = (1', 2' - 4', 5'),\ B_1 B'_1 = (3', 4' - 6', 1') \text{ and } C_1 C'_1$$

are concurrent. (Desargues.) For a similar reason the lines

$$A_2 A'_2 = (2', 3' - 5', 6'),\ B_2 B'_2 = (1', 2' - 4', 5'), \text{ and } C_2 C'_2$$

are concurrent; likewise the lines

$$A_3 A'_3 = (3', 4' - 6', 1'),\ B_3 B'_3 = (2', 3' - 5', 6'), \text{ and } C_3 C'_3.$$

But the triangles $C_1 C_2 C_3$ and $C'_1 C'_2 C'_3$ also fulfill the condition for Desargues' theorem, and therefore the lines

$$C_1 C'_1,\ C_2 C'_2, \text{ and } C_3 C'_3$$

are concurrent. Accordingly, the lines

$$(1', 2' - 4', 5'),\ (2', 3' - 5', 6'), \text{ and } (3', 4' - 6', 1')$$

are concurrent and the lines $1'$, $2'$, $3'$, $4'$, $5'$, $6'$ circumscribe a conic section. But this is the condition that the lines 1, 2, 3, 4, 5, 6 should also circumscribe a conic section.

24. *Theorem.—If Q is a tricuspidal quartic the three cuspidal tangents meet in one and the same point.* (See figure 28.)

To a tricuspidal quartic corresponds a conic tangent to each of the three sides of the triangle $S_1 S_2 S_3$. If the three points of contact on the sides s_1, s_2, s_3 be denoted by L, M, and N, respectively, then to the lines $S_1 L$, $S_2 M$, and $S_3 N$ correspond the three cuspidal tangents. But $S_1 L$, $S_2 M$, and $S_3 N$ are three concurrent lines (Brianchon), and therefore the three corresponding lines, the three cuspidal tangents, are concurrent.

25.—*Theorem.—If Q has three bitangents, their points of intersection may be joined with the three nodes, one with each node, in such a way that the three joining lines pass through one and the same point.* (See figure 29.) *If Q has four bitangents their points of intersection in sets of three may be joined with the nodes in such a way that the joining lines form a complete quadrangle of which each pair of opposite sides intersects in a node.* (See figure 30.)

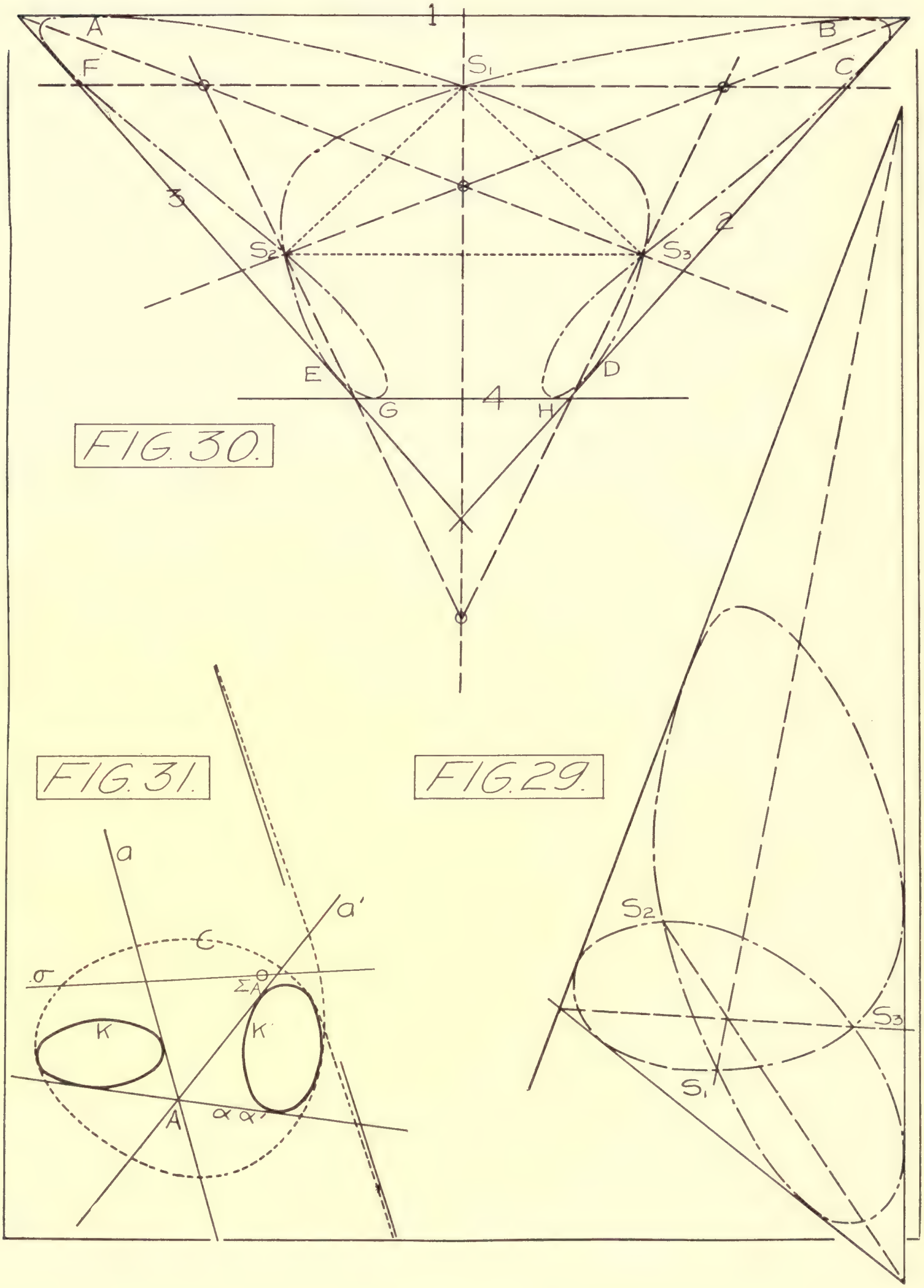
Each bitangent of Q corresponds by the quadratic involutory transformation to a conic passing through the three nodes, S_1, S_2, S_3, and having double contact with the Σ conic. Denote the four bitangents to Q by 1, 2, 3, 4; their points of contact on Q by $A, B; C, D; E, F;$ and G, H, respectively; and their corresponding conics through S_1, S_2, S_3, by I, II, III, and IV, respectively. Denote the fourth intersection of I and II by U; of II and III by V; of III and I by W; of I and IV by X; of II and IV by Y; and of III and IV by Z.

Consider first the case of three bitangents, 1, 2, 3. Then the lines

$$\Sigma_A \Sigma_B, \quad \Sigma_C \Sigma_D,$$

and one side of the triangle $S_1 S_2 S_3$, say $S_2 S_3$, and $S_1 U$ all pass through one and the same point and form a harmonic pencil.* Similarly, for the lines

* See Salmon, *Conic Sections*, § 263.



1
A
B
F
C
S₁
3
2
S₂
S₃
E
D
4
G
H
FIG. 30.
FIG. 31.
FIG. 29.
a
a′
σ
ε
Σ_A
K
K′
A
α α′
S₂
S₃
S₁

Figs. 29–31

$$\Sigma_C\,\Sigma_D,\ \Sigma_E\,\Sigma_F,\ S_3\,S_1, \text{ and } S_2\,V;$$

likewise, for the lines

$$\Sigma_E\,\Sigma_F,\ \Sigma_A\,\Sigma_B,\ S_1\,S_2, \text{ and } S_3\,W.$$

From this it follows that the lines

$$S_1\,U,\ S_2\,V, \text{ and } S_3\,W$$

are concurrent. For on $S_3\,W$ the lines

$$\Sigma_A\,\Sigma_B,\ \Sigma_C\,\Sigma_D,\ S_2\,S_3, \text{ and } S_1\,U$$

determine four harmonic points. Likewise

$$\Sigma_E\,\Sigma_F,\ \Sigma_C\,\Sigma_D,\ S_1\,S_3, \text{ and } S_2\,V$$

determine four harmonic points on $S_3\,W$. But on $S_3\,W$, $\Sigma_C\,\Sigma_D$ in each case determines the same point; $S_2\,S_3$ and $S_1\,S_3$ both determine S_3; $\Sigma_A\,\Sigma_B$ and $\Sigma_E\,\Sigma_F$ determine the same point since the lines

$$\Sigma_E\,\Sigma_F,\ \Sigma_A\,\Sigma_B,\ S_1\,S_2, \text{ and } S_3\,W$$

are concurrent. Therefore, $S_1\,U$ and $S_2\,V$ determine the same point on $S_3\,W$; that is, $S_1\,U$, $S_2\,V$, and $S_3\,W$ are concurrent.

But if these three lines are concurrent, the lines to which they correspond in involution must be concurrent. That is, the lines

$$(S_1 - 1, 2),\ (S_2 - 2, 3),\ (S_3 - 3, 1)$$

pass through one and the same point.

Consider now the case of four bitangents to Q, 1, 2, 3, 4. Proceeding as before, we find the following sets of harmonic lines:

For I, II, and III—

I and II	II and III	III and I
$S_2\,S_3$	$S_3\,S_1$	$S_1\,S_2$
$S_1\,U$	$S_2\,V$	$S_3\,W$
$\Sigma_A\,\Sigma_B$	$\Sigma_C\,\Sigma_D$	$\Sigma_E\,\Sigma_F$
$\Sigma_C\,\Sigma_D$	$\Sigma_E\,\Sigma_F$	$\Sigma_A\,\Sigma_B$

For I, II, and IV—

I and II	II and IV	IV and I
$S_2\,S_3$	$S_1\,S_2$	$S_1\,S_3$
$S_1\,U$	$S_3\,Y$	$S_2\,X$
$\Sigma_A\,\Sigma_B$	$\Sigma_C\,\Sigma_D$	$\Sigma_G\,\Sigma_H$
$\Sigma_C\,\Sigma_D$	$\Sigma_G\,\Sigma_H$	$\Sigma_A\,\Sigma_B$

For I, III, and IV—

I and III	III and IV	I and IV
$S_1\,S_2$	$S_2\,S_3$	$S_1\,S_3$
$S_3\,W$	$S_1\,Z$	$S_2\,X$
$\Sigma_E\,\Sigma_F$	$\Sigma_E\,\Sigma_F$	$\Sigma_G\,\Sigma_H$
$\Sigma_A\,\Sigma_B$	$\Sigma_G\,\Sigma_H$	$\Sigma_A\,\Sigma_B$

For II, III, and IV—

II and III	III and IV	IV and II
$S_3\,S_1$	$S_2\,S_3$	$S_1\,S_2$
$S_2\,V$	$S_1\,Z$	$S_3\,Y$
$\Sigma_C\,\Sigma_D$	$\Sigma_E\,\Sigma_F$	$\Sigma_C\,\Sigma_D$
$\Sigma_E\,\Sigma_F$	$\Sigma_G\,\Sigma_H$	$\Sigma_G\,\Sigma_H$

From this we obtain the following four sets of three concurrent lines:

$$\left.\begin{matrix} S_1\,U \\ S_2\,V \\ S_3\,W \end{matrix}\right\} \quad \left.\begin{matrix} S_1\,U \\ S_2\,X \\ S_3\,Y \end{matrix}\right\} \quad \left.\begin{matrix} S_1\,Z \\ S_2\,X \\ S_3\,W \end{matrix}\right\} \quad \left.\begin{matrix} S_1\,Z \\ S_2\,V \\ S_3\,Y \end{matrix}\right\}$$

The configuration determined by these six lines is a complete quadrangle of which each pair of opposite sides intersects in a node of Q. Each set of three concurrent lines, one of which passes through each involution center, corresponds in involution to another set of three concurrent lines, one of which passes through each involution center. Therefore, the figure corresponding in involution to a complete quadrangle of which each pair of opposite sides intersects in an involution center is another complete quadrangle of the same character; and the lines $S_1\,U$, $S_2\,V$, $S_3\,W$, $S_1\,Z$, $S_2\,X$, $S_3\,Y$ correspond in involution to lines joining the points of intersection of the bitangents of Q to the nodes.

26. *Degenerate Cases.*—In general, the pencils κ and κ' have no self-corresponding rays. That is to say, a tangent common to κ and κ', considered as a ray of κ, corresponds in general to some other ray of κ', and considered as a ray of κ', to some other ray of κ. But it may happen that of the three pairs of corresponding rays of κ and κ' given to construct Q (§ 6), one, two, or all three may be self-corresponding. Also, in general, the conics κ and κ' are distinct. But the two pencils of the second order may envelope the same base conic; and in that case, too, there may or may not be self-corresponding rays. In all these special cases both Q and Σ assume degenerate forms.

A. If κ and κ' have one self-corresponding ray, $\alpha\,\alpha'$, Q consists of a cubic C and the ray $\alpha\,\alpha'$; Σ consists of a line σ and the ray $\alpha\,\alpha'$; points of Q on $\alpha\,\alpha'$ correspond to points of Σ on σ and points of Q on C to points of Σ on $\alpha\,\alpha'$. (See figure 31.)

Since α and α' coincide, the four tangents from Σ_A to κ and κ' meet $\alpha\,\alpha'$ in four points of Σ, showing at once that all the points of $\alpha\,\alpha'$ are points of Σ, since more than two of the points of $\alpha\,\alpha'$ are points of Σ. That is, Σ degenerates into two straight lines, one of which is $\alpha\,\alpha'$.

If from points of Σ as Σ_D, Σ_E, Σ_F, etc., on $\alpha\,\alpha'$ we draw tangents to κ and κ', such tangents must meet the lines d', d, e', e, f', f, etc., in points of Q. But since always $\alpha\,\alpha'$ itself will be a tangent from such points both to κ and κ', it follows that half of the points so obtained will lie on $\alpha\,\alpha'$. $\alpha\,\alpha'$ is then a part of Q. The remaining part is, of course, a cubic C.

The construction shows at once the correspondence between the different parts of Q and Σ.

B. If κ and κ' have two self-corresponding rays, $\alpha\,\alpha'$ and $\beta\,\beta'$, Q consists of the two lines $\alpha\,\alpha'$ and $\beta\,\beta'$ and a conic λ; Σ consists of the two lines $\alpha\,\alpha'$ and $\beta\,\beta'$; points of Q on $\alpha\,\alpha'$ correspond to points of Σ on $\beta\,\beta'$ and points of Q on $\beta\,\beta'$ to points of Σ on $\alpha\,\alpha'$, while the points of Q on the conic λ all have their corresponding points of Σ at the intersection of $\alpha\,\alpha'$ and $\beta\,\beta'$.

C. If κ and κ' have three self-corresponding rays, $\alpha\,\alpha'$, $\beta\,\beta'$, and $\gamma\,\gamma'$, Q consists of the four tangents common to κ and κ'; Σ consists of the three vertices of the triangle formed by $\alpha\,\alpha'$, $\beta\,\beta'$, and $\gamma\,\gamma'$. Points of Q on one of the three lines

$\alpha\ \alpha'$, $\beta\ \beta'$, or $\gamma\ \gamma'$ correspond to the opposite vertex of the triangle formed by those lines. Points of Σ corresponding to points of Q on the fourth tangent common to κ and κ' are indeterminate.

To prove that the fourth tangent common to κ and κ' is a part of Q, find the point P on a for which a is a ray π of κ. p and p' must coincide. Suppose they do not. p, considered as a ray ρ of κ, determines a point R of Q, and p', considered as a ray ξ' of κ', determines a point X of Q. None of the points P, R, and X can lie on any of the rays $\alpha\ \alpha'$, $\beta\ \beta'$, $\gamma\ \gamma'$, since every point of Q on a self-corresponding ray is determined by that ray. Since Q is of the fourth degree, P, R, and X must lie in one and the same straight line. This can happen only if p and p' coincide.

D. The two projective pencils of the second order, κ' and κ'', envelope the same base conic κ.

(1) If there are no self-corresponding rays, Σ degenerates into two straight lines σ_1 and σ_2 tangent to κ, and Q into these same two lines and a conic section λ. The construction shows that points of Q on σ_1 correspond to points of Σ on σ_2, and points of Q on σ_2 to points of Σ on σ_1, while points of Q on λ all have their corresponding points of Σ at the intersection of σ_1 and σ_2. The conic λ is tangent to κ at the points of tangency of σ_1 and σ_2.

Here a coincides with α', a' with α, b with β', b' with β, etc. It follows that the line $(a, \beta - \alpha', \beta')$ coincides with $(b', \alpha' - b, \alpha)$; $(a, \gamma - \alpha', \gamma')$ with $(c', \alpha' - c, \alpha)$; and $(b, \gamma - b', \gamma')$ with $(c', \beta' - c, \beta)$. These are three lines joining the opposite vertices of a hexagon circumscribed to a conic section and are therefore concurrent. (Brianchon.) Therefore, Σ_A, Σ_B, Σ_C all coincide. The tangents from Σ_A, Σ_B, Σ_C, etc., to κ determine points of Σ on α, α', β, β', γ, γ', etc., and points of Q on a, a', b, b', c, c', etc. Because of the coincidences mentioned all these points lie on two lines, σ_1 and σ_2, the tangents from $\Sigma_{ABC,\ etc.}$, to κ.

(2) If κ' and κ'' have one self-corresponding ray $\alpha\ \alpha'$, $\alpha\ \alpha'$ coincides with σ_1 or σ_2, and Σ_A is indeterminate. In other respects Q and Σ are as described in (1).

(3) If κ' and κ'' have two self-corresponding rays $\alpha\ \alpha'$ and $\beta\ \beta'$, $\alpha\ \alpha'$ and $\beta\ \beta'$ are the lines σ_1 and σ_2, and Σ_A and Σ_B are indeterminate. In other respects Q and Σ are as described in (1).

(4) If κ' and κ'' have three self-corresponding rays, $\alpha\ \alpha'$, $\beta\ \beta'$, and $\gamma\ \gamma'$, everything is indeterminate.

27. *The Unicursal Curve of the Fourth Class.*—It is of interest to apply the principle of duality to the results of this chapter, considering then the unicursal curve Q' defined as the envelope of lines joining corresponding points in two projective point rows of the second order, κ and κ'. It has, in general, three bitangents and four nodes and is of the sixth order and fourth class. (Cf. §§ 1, 2, 20, 25.) Given three pairs of corresponding points in two projective point rows, we can construct the curve (cf. § 6) and establish properties entirely analogous to all those of the curve of the fourth order. The more interesting and important ones may be noted.

Every ray p of the rays enveloping Q' has its corresponding ray of perspectivity Σ'_p. The ensemble of rays Σ'_p envelope a curve Σ' of the second class. (Cf. §§ 7 and 13.)

Any ray a joining corresponding points A_1 and A'_1 has a second point A_2 in common with κ and a second point A'_2 in common with κ'. The lines joining the points of intersection of the ray Σ'_a and κ with A'_2 are rays enveloping Q'; with A_1, are rays enveloping Σ'. Likewise, the lines joining the points of intersection of the ray Σ'_a and κ' with A_2 are rays enveloping Q'; with A'_1, are rays enveloping Σ'. The ray tangent to Σ' which joins A_1 (or A'_1) with one point of intersection of Σ'_a and κ (or κ') corresponds to the ray tangent to Q' which joins A'_2 (or A_2) with the other point of intersection of Σ'_a and κ (or κ'). (Cf. § 11.)

The common rays of the sets enveloping Q' and Σ' pass through the points of intersection of κ and κ', two through each point. (Cf. § 14.)

The points A_2 and A'_1, A'_2 and A_1 determine an involution of points on a in which the point (a, Σ'_a) corresponds to the point of tangency of a on Q'. (Cf. § 19.)

The six points of Q' other than the points of tangency which lie on the three bitangents to Q' all lie on one and the same conic section. (Cf. § 21.)

The six points of tangency on the three bitangents to Q' all lie on one and the same conic section. (Cf. § 23.)

If the two points of tangency coincide on each bitangent the three points of tangency all lie on one and the same straight line. (Cf. § 24.)

If Q' has three nodes the lines joining them intersect the three bitangents, one each bitagent, in such a way that the three points of intersection lie on one and the same straight line. If Q' has four nodes the lines joining them in sets of three intersect the bitangents in such a way that the points of intersection form a complete quadrilateral of which each pair of opposite vertices lies on a bitangent. (Cf. § 25.)

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